Big Step Semantics

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Objectives

- Describe the components of a big step semantic rule.
- Use semantic rules to document the meaning of simple programming language.
- Explain the correspondence between big step semantics and the `eval` function.

Grammar for Simple Imperative Programming Language

The Language

$$
\begin{array}{rcl}
S & ::= & \texttt{skip} \\
 & | & u := A \\
 & | & S_1 ; S_2 \\
 & | & \texttt{if } B \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ fi} \\
 & | & \texttt{while } B \texttt{ do } S_1 \texttt{ od} \\
B & ::= & E \sim E \\
 & | & \texttt{true} \mid \texttt{false} \\
E & ::= & u \\
 & | & int \\
 & | & E \oplus E
\end{array}
$$

- Let u be a possibly subscripted variable.
- E represents arithmetic expressions, $\oplus$ is an arithmetic operator.

The Downarrow Notation

- In small step semantics we use the $\rightarrow$ to represent one step of computations.
- In big step semantics we use $\Downarrow$ to represent an entire evaluation.

Statements

$$< S, \sigma > \Downarrow \sigma'$$

Expressions

$$< E, \sigma > \Downarrow_e v$$

Booleans

$$< B, \sigma > \Downarrow_b b$$

Expressions

Integers

$$\frac{}{< i, \sigma >\Downarrow_e i}\ \textsc{Const}$$

if i is an integer.

Variables

$$\frac{}{< u, \sigma >\Downarrow_e v}\ \textsc{Var}$$

if $u := v \in \sigma$.

Operations

$$\frac{< e_1, \sigma >\Downarrow_e v_1 \qquad < e_2, \sigma >\Downarrow_e v_2}{< e_1 \oplus e_2, \sigma >\Downarrow_e v_1 \oplus v_2}\ \textsc{Arith}$$

Here $\oplus$ represents typical binary operations like $+, -, \times$, etc.

Boolean Expressions

Booleans

$$\frac{}{< b, \sigma >\Downarrow_b b}\ \textsc{Const}$$

if b is a boolean.

Variables

$$\frac{}{< u, \sigma >\Downarrow_b v}\ \textsc{Var}$$

if $u := v \in \sigma$.

Relational Operators

$$\frac{< e_1, \sigma >\Downarrow_e v_1 \qquad < e_2, \sigma >\Downarrow_e v_2}{< e_1 \sim e_2, \sigma >\Downarrow_b v_1 \sim v_2}\ \textsc{Rel}$$

Here $\sim$ represents the binary relational operations $=, \leq, \geq, \neq, \geq, \leq$, etc.

Skip and Assignment

$$\frac{}{< \texttt{skip}\,, \sigma >\Downarrow \sigma}\ \textsc{Skip}$$

$$\frac{< e, \sigma >\Downarrow_e v}{< x := e, \sigma >\Downarrow \sigma[x := v]}\ \textsc{Assign}$$

Skip and Assignment

$$\frac{}{< \texttt{skip}\,, \sigma >\Downarrow \sigma}\ \textsc{Skip}$$

$$\frac{< e, \sigma >\Downarrow_e v}{< x := e, \sigma >\Downarrow \sigma[x := v]}\ \textsc{Assign}$$

Next is sequencing. See if you can guess what the rule looks like.

Sequencing

$$\frac{< S_1, \sigma > \Downarrow \sigma' \qquad < S_2, \sigma' > \Downarrow \sigma''}{< S_1; S_2, \sigma > \Downarrow \sigma''}\ \textsc{Seq}$$

Sequencing

$$\frac{< S_1, \sigma > \Downarrow \sigma' \qquad < S_2, \sigma' > \Downarrow \sigma''}{< S_1; S_2, \sigma > \Downarrow \sigma''}\ \textsc{Seq}$$

Next is **`if`** . There are two rules for this. See if you can guess what the rules looks like.

If Statements

$$\frac{< B, \sigma > \Downarrow_b \texttt{true} \qquad < S_1, \sigma > \Downarrow \sigma'}{< \texttt{if } B \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ fi} , \sigma > \Downarrow \sigma'}\ \textsc{If}_1$$

$$\frac{< B, \sigma > \Downarrow_b \texttt{false} \qquad < S_2, \sigma > \Downarrow \sigma'}{< \texttt{if } B \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ fi} , \sigma > \Downarrow \sigma'}\ \textsc{If}_2$$

If Statements

$$\frac{< B, \sigma > \Downarrow_b \texttt{true} \qquad < S_1, \sigma > \Downarrow \sigma'}{< \texttt{if } B \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ fi} , \sigma > \Downarrow \sigma'}\ \textsc{If}_1$$

$$\frac{< B, \sigma > \Downarrow_b \texttt{false} \qquad < S_2, \sigma > \Downarrow \sigma'}{< \texttt{if } B \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ fi} , \sigma > \Downarrow \sigma'}\ \textsc{If}_2$$

Next is **`while`** . There are two rules for this. See if you can guess what the rules looks like. The second one uses induction!

While Statements

$$\frac{< B, \sigma >\Downarrow_b \texttt{false}}{< \texttt{while}\ B\ \texttt{do}\ S\ \texttt{od}\ , \sigma >\Downarrow \sigma}\ \textsc{While}_1$$

$$\frac{< B, \sigma >\Downarrow_b \texttt{true} \qquad < S, \sigma >\Downarrow \sigma' \qquad < \texttt{while}\ B\ \texttt{do}\ S\ \texttt{od}\ , \sigma' >\Downarrow \sigma''}{< \texttt{while}\ B\ \texttt{do}\ S\ \texttt{od}\ , \sigma >\Downarrow \sigma''}\ \textsc{While}_2$$

Proof Trees

- To show the effect of a program, we need to build proof trees.
- Let $\sigma = \{x := 3, y := 4\}$.
- We want to prove that $2 \times y + 9 \times x = 35$.

Here is what we want to evaluate. What kind of expression is this?

$$\frac{}{< 2 \times y + 9 \times x, \sigma >\Downarrow_e 35}$$

Proof Trees

- To show the effect of a program, we need to build proof trees.
- Let $\sigma = \{x := 3, y := 4\}$.
- We want to prove that $2 \times y + 9 \times x = 35$.

Because of precedence rules, we evaluate the + last.

$$\frac{\frac{}{< 2 \times y, \sigma >\Downarrow_e 8} \qquad \frac{}{< 9 \times x, \sigma >\Downarrow_e 27}}{< 2 \times y + 9 \times x, \sigma >\Downarrow_e 35}\ \textsc{Arith}$$

Proof Trees

- To show the effect of a program, we need to build proof trees.
- Let $\sigma = \{x := 3, y := 4\}$.
- We want to prove that $2 \times y + 9 \times x = 35$.

We go up one more level, and then we are done.

$$\frac{\frac{\frac{}{< 2, \sigma >\Downarrow_e 2}\ \textsc{Const} \qquad \frac{}{< y, \sigma >\Downarrow_e 4}\ \textsc{Var}}{< 2 \times y, \sigma >\Downarrow_e 8}\ \textsc{Arith} \qquad \frac{\frac{}{< 9, \sigma >\Downarrow_e 9}\ \textsc{Const} \qquad \frac{}{< x, \sigma >\Downarrow_e 3}\ \textsc{Var}}{< 9 \times x, \sigma >\Downarrow_e 27}\ \textsc{Arith}}{< 2 \times y + 9 \times x, \sigma >\Downarrow_e 35}\ \textsc{Arith}$$

Statement Proof Tree

- Let $\sigma = \{x := 10, y := 20\}$.
- Let $\sigma' = \{x := 10, y := 20, m := 20\}$.

Here is an example that will use all three versions of $\Downarrow$.

$$\dfrac{}{< \texttt{if}\ x > y\ \texttt{then}\ m := x\ \texttt{else}\ m := 2 \times x\ \texttt{fi}\ , \sigma > \Downarrow \sigma'}$$

Can you figure out what this tree should look like?

Statement Proof Tree

- Let $\sigma = \{x := 10, y := 20\}$.
- Let $\sigma' = \{x := 10, y := 20, m := 20\}$.

$$\dfrac{\dfrac{}{< x > y, \sigma > \Downarrow_b \texttt{false}}\ \textsc{Rel} \qquad \dfrac{}{< m := 2 \times x, \sigma > \Downarrow \sigma'}\ \textsc{Assign}}{< \texttt{if}\ x > y\ \texttt{then}\ m := x\ \texttt{else}\ m := 2 \times x\ \texttt{fi}\ , \sigma > \Downarrow \sigma'}\ \textsc{If}_2$$

Statement Proof Tree

- Let $\sigma = \{x := 10, y := 20\}$.
- Let $\sigma' = \{x := 10, y := 20, m := 20\}$.

$$\dfrac{\dfrac{\dfrac{}{< x, \sigma > \Downarrow_e 10}\ \textsc{Var} \quad \dfrac{}{< y, \sigma > \Downarrow_e 20}\ \textsc{Var}}{< x > y, \sigma > \Downarrow_b \texttt{false}}\ \textsc{Rel} \qquad \dfrac{\dfrac{}{< 2, \sigma > \Downarrow_e 2}\ \textsc{Const} \quad \dfrac{}{< x, \sigma > \Downarrow_e 10}\ \textsc{Var}}{< m := 2 \times x, \sigma > \Downarrow \sigma'}\ \textsc{Assign}}{< \texttt{if}\ x > y\ \texttt{then}\ m := x\ \texttt{else}\ m := 2 \times x\ \texttt{fi}\ , \sigma > \Downarrow \sigma'}\ \textsc{If}_2$$

Connecting to Interpreters

- The $\Downarrow$ is really just `eval` that you already know and love.
- The σ is just the `env` parameter.