# Lambda Calculus Examples

Dr. Mattox Beckman

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
DEPARTMENT OF COMPUTER SCIENCE



## Objectives

You should be able to ...

Here are some examples!

- Perform a beta-reduction.
- Detect $\alpha$-capture and use $\alpha$-renaming to avoid it.
- Normalize any given $\lambda$-calculus term.



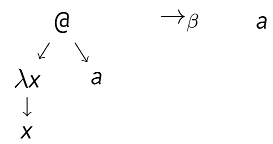
## Examples

$(\lambda x.x)\ a$
$(\lambda x.x\ x)\ a$
$(\lambda x.y\ x)\ a$
$(\lambda x.\lambda a.x)\ a$
$(\lambda x.\lambda x.x)\ a$
$(\lambda x.(\lambda y.y)\ x)\ a$

Objectives ○ Examples ●○○○○○○○○

## Examples

$(\lambda x.x)\ a \quad \to_\beta \quad a$
$(\lambda x.x\ x)\ a$
$(\lambda x.y\ x)\ a$
$(\lambda x.\lambda a.x)\ a$
$(\lambda x.\lambda x.x)\ a$
$(\lambda x.(\lambda y.y)\ x)\ a$



# Examples

$(\lambda x.x)\ a \quad \to_\beta \quad a$

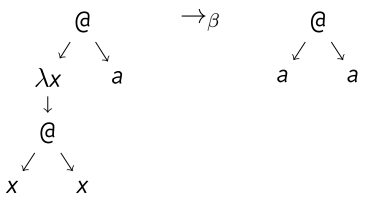
$(\lambda x.x\ x)\ a \quad \to_\beta \quad a\ a$

$(\lambda x.y\ x)\ a$

$(\lambda x.\lambda a.x)\ a$

$(\lambda x.\lambda x.x)\ a$

$(\lambda x.(\lambda y.y)\ x)\ a$






# Examples

$(\lambda x.x)\ a \quad \to_\beta \quad a$

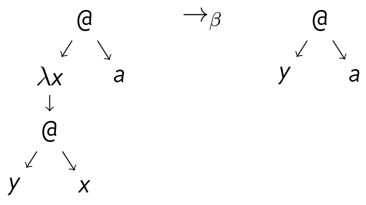
$(\lambda x.x\ x)\ a \quad \to_\beta \quad a\ a$

$(\lambda x.y\ x)\ a \quad \to_\beta \quad y\ a$

$(\lambda x.\lambda a.x)\ a$

$(\lambda x.\lambda x.x)\ a$

$(\lambda x.(\lambda y.y)\ x)\ a$






# Examples

$(\lambda x.x)\ a \quad \to_\beta \quad a$

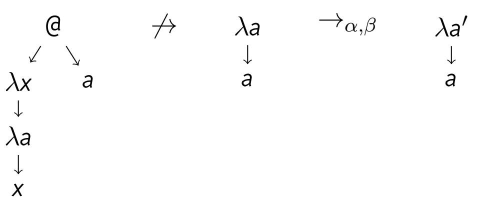
$(\lambda x.x\ x)\ a \quad \to_\beta \quad a\ a$

$(\lambda x.y\ x)\ a \quad \to_\beta \quad y\ a$

$(\lambda x.\lambda a.x)\ a \quad \to_\alpha \quad (\lambda x.\lambda a'.x) \quad \to_\beta \quad \lambda a'.a$

$(\lambda x.\lambda x.x)\ a$

$(\lambda x.(\lambda y.y)\ x)\ a$






# Examples

$(\lambda x.x)\ a \quad \to_\beta \quad a$

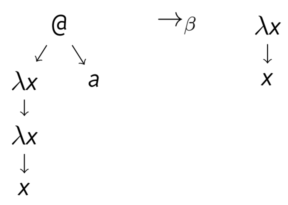
$(\lambda x.x\ x)\ a \quad \to_\beta \quad a\ a$

$(\lambda x.y\ x)\ a \quad \to_\beta \quad y\ a$

$(\lambda x.\lambda a.x)\ a \quad \to_\alpha \quad (\lambda x.\lambda a'.x) \quad \to_\beta \quad \lambda a'.a$

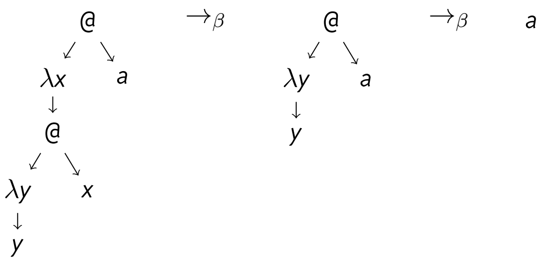
$(\lambda x.\lambda x.x)\ a \quad \to_\beta \quad \lambda x.x$

$(\lambda x.(\lambda y.y)\ x)\ a$



## Examples

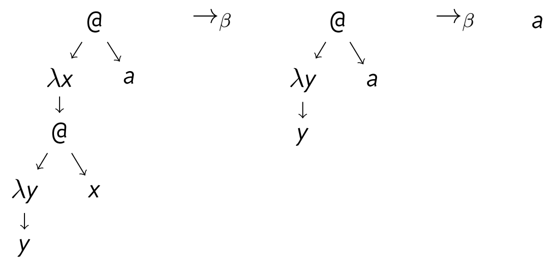
| | | | | |
|---|---|---|---|---|
| $(\lambda x.x)\ a$ | $\rightarrow_\beta$ | $a$ | | |
| $(\lambda x.x\ x)\ a$ | $\rightarrow_\beta$ | $a\ a$ | | |
| $(\lambda x.y\ x)\ a$ | $\rightarrow_\beta$ | $y\ a$ | | |
| $(\lambda x.\lambda a.x)\ a$ | $\rightarrow_\alpha$ | $(\lambda x.\lambda a'.x)$ | $\rightarrow_\beta$ | $\lambda a'.a$ |
| $(\lambda x.\lambda x.x)\ a$ | $\rightarrow_\beta$ | $\lambda x.x$ | | |
| $(\lambda x.(\lambda y.y)\ x)\ a$ | $\rightarrow_\beta$ | $(\lambda y.y)\ a$ | | |






## Examples

| | | | | |
|---|---|---|---|---|
| $(\lambda x.x)\ a$ | $\rightarrow_\beta$ | $a$ | | |
| $(\lambda x.x\ x)\ a$ | $\rightarrow_\beta$ | $a\ a$ | | |
| $(\lambda x.y\ x)\ a$ | $\rightarrow_\beta$ | $y\ a$ | | |
| $(\lambda x.\lambda a.x)\ a$ | $\rightarrow_\alpha$ | $(\lambda x.\lambda a'.x)$ | $\rightarrow_\beta$ | $\lambda a'.a$ |
| $(\lambda x.\lambda x.x)\ a$ | $\rightarrow_\beta$ | $\lambda x.x$ | | |
| $(\lambda x.(\lambda y.y)\ x)\ a$ | $\rightarrow_\beta$ | $(\lambda y.y)\ a \rightarrow_\beta\ a$ | | |






## $\alpha$ capture

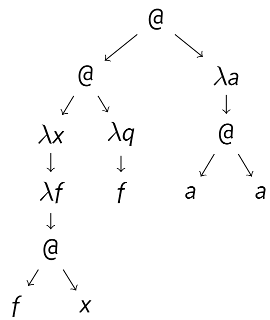
$$(\lambda x.\lambda a.x)\ a \rightarrow_\alpha \quad (\lambda x.\lambda a'.x) \quad \rightarrow_\beta \quad \lambda a'.a$$

- If a free occurrence of a variable gets placed under a $\lambda$ that binds it, this is called $\alpha$ capture.
- To resolve this, rename the *binder*.

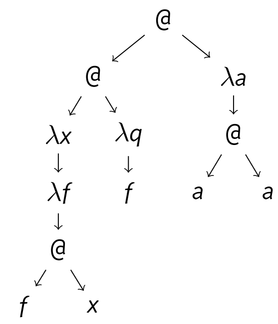


## Here's One for You to Try!

- Convert this tree into an equivalent $\lambda$ term.
- Identify the free variables.
- Simplify it by performing as many $\beta$ reductions (and necessary $\alpha$ renamings) as possible.



## Solution

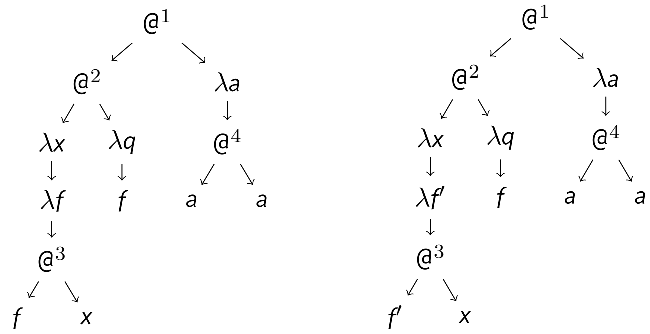


$(\lambda xf.fx)(\lambda q.f)(\lambda a.aa)$

- There is one free variable ....



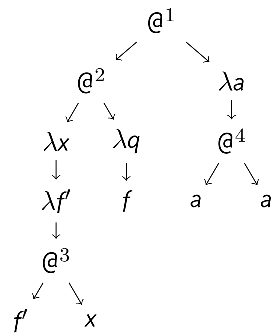
## Solution, Step 1



$(\lambda xf.fx)(\lambda q.f)(\lambda a.aa) \quad \to_\alpha \quad (\lambda xf'.f'x)(\lambda q.f)(\lambda a.aa)$



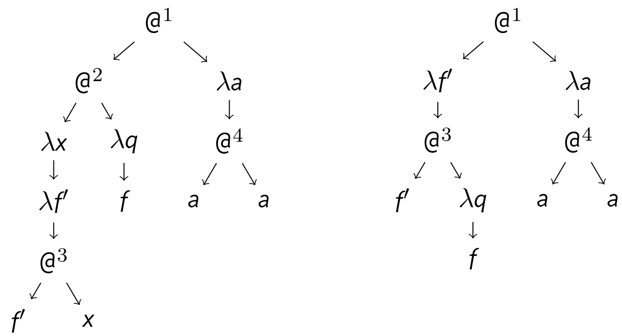
## Solution, Step 2



$(\lambda xf'.f'x)(\lambda q.f)(\lambda a.aa) \quad \to_\beta$

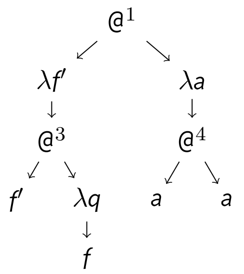


## Solution, Step 2



$(\lambda xf'.f'x)(\lambda q.f)(\lambda a.aa) \quad \to_\beta \quad (\lambda f'.f'(\lambda q.f))(\lambda a.aa)$

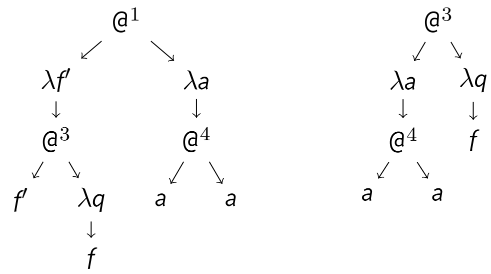
## Solution, Step 3



$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \to_\beta$$



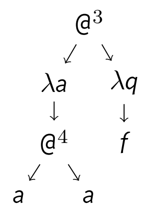
## Solution, Step 3



$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \to_\beta \quad (\lambda a.aa)(\lambda q.f)$$



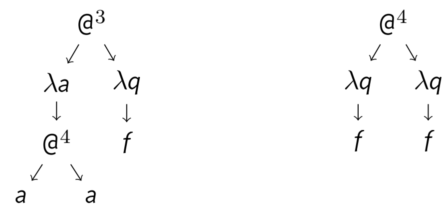
## Solution, Step 4



$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \to_\beta$$



## Solution, Step 4



$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \to_\beta \quad (\lambda q.f)(\lambda q.f)$$

## Solution, Step 5

$$@^4$$
$$\swarrow \quad \searrow$$
$$\lambda q \qquad \lambda q$$
$$\downarrow \qquad \downarrow$$
$$f \qquad f$$

$$(\lambda q.f)(\lambda q.f) \quad \rightarrow_\beta$$

Objectives ○ Examples ○○○○○○○○●

## Solution, Step 5

$$@^4 \qquad\qquad f$$
$$\swarrow \quad \searrow$$
$$\lambda q \qquad \lambda q$$
$$\downarrow \qquad \downarrow$$
$$f \qquad f$$

$$(\lambda q.f)(\lambda q.f) \quad \rightarrow_\beta \quad f$$