Lambda Calculus Examples

Dr. Mattox Beckman

University of Illinois at Urbana-Champaign
Department of Computer Science

Objectives

You should be able to ...

Here are some examples!

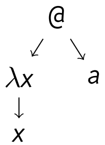
- Perform a beta-reduction.
- Detect α-capture and use α-renaming to avoid it.
- Normalize any given λ-calculus term.

Examples

$$(\lambda x.x)\ a$$
$$(\lambda x.x\ x)\ a$$
$$(\lambda x.y\ x)\ a$$
$$(\lambda x.\lambda a.x)\ a$$
$$(\lambda x.\lambda x.x)\ a$$
$$(\lambda x.(\lambda y.y)\ x)\ a$$

Examples

$$(\lambda x.x)\ a \quad \rightarrow_\beta \quad a$$
$$(\lambda x.x\ x)\ a$$
$$(\lambda x.y\ x)\ a$$
$$(\lambda x.\lambda a.x)\ a$$
$$(\lambda x.\lambda x.x)\ a$$
$$(\lambda x.(\lambda y.y)\ x)\ a$$



$$\rightarrow_\beta \quad a$$

Examples

$$(\lambda x.x)\ a \quad \rightarrow_\beta \quad a$$

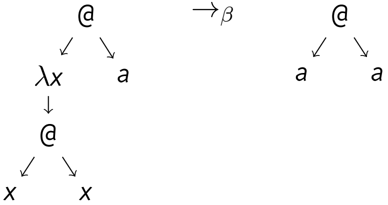
$$(\lambda x.x\ x)\ a \quad \rightarrow_\beta \quad a\ a$$

$$(\lambda x.y\ x)\ a$$

$$(\lambda x.\lambda a.x)\ a$$

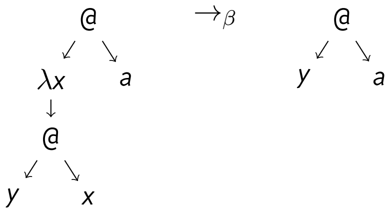
$$(\lambda x.\lambda x.x)\ a$$

$$(\lambda x.(\lambda y.y)\ x)\ a$$



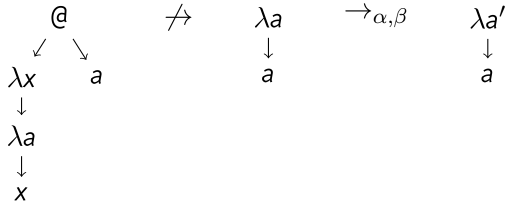
Examples

$$(\lambda x.x)\ a \quad \rightarrow_\beta \quad a$$
$$(\lambda x.x\ x)\ a \quad \rightarrow_\beta \quad a\ a$$
$$(\lambda x.y\ x)\ a \quad \rightarrow_\beta \quad y\ a$$
$$(\lambda x.\lambda a.x)\ a$$
$$(\lambda x.\lambda x.x)\ a$$
$$(\lambda x.(\lambda y.y)\ x)\ a$$



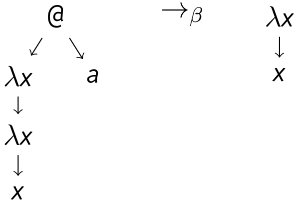
Examples

$$
\begin{array}{lllll}
(\lambda x.x)\ a & \rightarrow_\beta & a & & \\
(\lambda x.x\ x)\ a & \rightarrow_\beta & a\ a & & \\
(\lambda x.y\ x)\ a & \rightarrow_\beta & y\ a & & \\
(\lambda x.\lambda a.x)\ a & \rightarrow_\alpha & (\lambda x.\lambda a'.x) & \rightarrow_\beta & \lambda a'.a \\
(\lambda x.\lambda x.x)\ a & & & & \\
(\lambda x.(\lambda y.y)\ x)\ a & & & &
\end{array}
$$



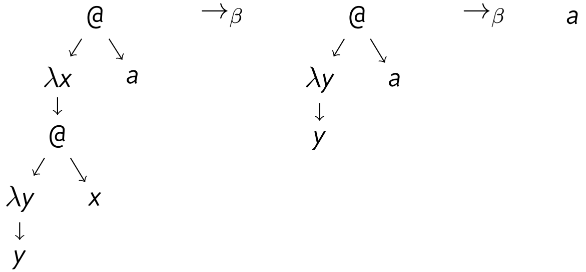
Examples

$$
\begin{array}{lllll}
(\lambda x.x)\ a & \to_\beta & a & & \\
(\lambda x.x\ x)\ a & \to_\beta & a\ a & & \\
(\lambda x.y\ x)\ a & \to_\beta & y\ a & & \\
(\lambda x.\lambda a.x)\ a & \to_\alpha & (\lambda x.\lambda a'.x) & \to_\beta & \lambda a'.a \\
(\lambda x.\lambda x.x)\ a & \to_\beta & \lambda x.x & & \\
(\lambda x.(\lambda y.y)\ x)\ a & & & &
\end{array}
$$



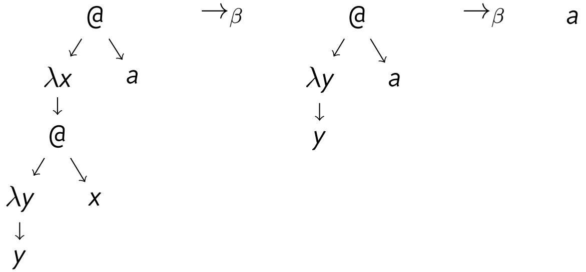
Examples

$(\lambda x.x)\ a$	$\rightarrow_\beta$	a		
$(\lambda x.x\ x)\ a$	$\rightarrow_\beta$	$a\ a$		
$(\lambda x.y\ x)\ a$	$\rightarrow_\beta$	$y\ a$		
$(\lambda x.\lambda a.x)\ a$	$\rightarrow_\alpha$	$(\lambda x.\lambda a'.x)$	$\rightarrow_\beta$	$\lambda a'.a$
$(\lambda x.\lambda x.x)\ a$	$\rightarrow_\beta$	$\lambda x.x$		
$(\lambda x.(\lambda y.y)\ x)\ a$	$\rightarrow_\beta$	$(\lambda y.y)\ a$		



Examples

$$
\begin{array}{llllll}
(\lambda x.x)\ a & \to_\beta & a & & & \\
(\lambda x.x\ x)\ a & \to_\beta & a\ a & & & \\
(\lambda x.y\ x)\ a & \to_\beta & y\ a & & & \\
(\lambda x.\lambda a.x)\ a & \to_\alpha & (\lambda x.\lambda a'.x) & \to_\beta & \lambda a'.a \\
(\lambda x.\lambda x.x)\ a & \to_\beta & \lambda x.x & & & \\
(\lambda x.(\lambda y.y)\ x)\ a & \to_\beta & (\lambda y.y)\ a \to_\beta\ a & & &
\end{array}
$$



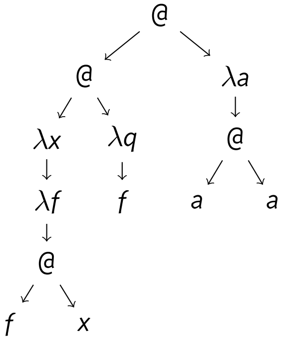
α capture

$$(\lambda x.\lambda a.x)\, a \rightarrow_\alpha \quad (\lambda x.\lambda a'.x) \quad \rightarrow_\beta \quad \lambda a'.a$$

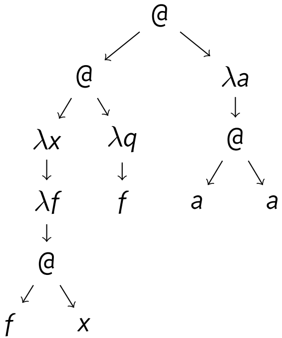
- If a free occurrence of a variable gets placed under a λ that binds it, this is called α capture.
- To resolve this, rename the *binder*.

Here's One for You to Try!

- Convert this tree into an equivalent λ term.
- Identify the free variables.
- Simplify it by performing as many β reductions (and necessary α renamings) as possible.



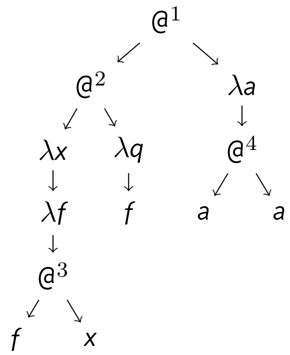
Solution

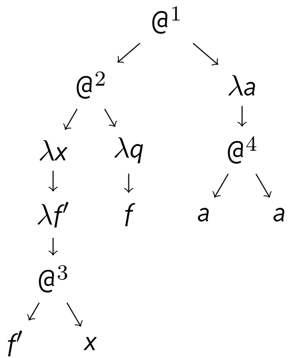


$$(\lambda x f.fx)(\lambda q.f)(\lambda a.aa)$$

- There is one free variable ....

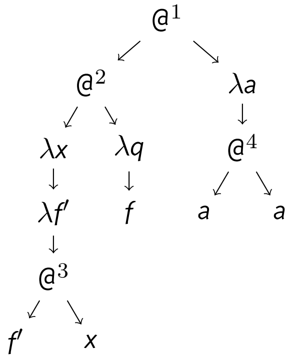
Solution, Step 1





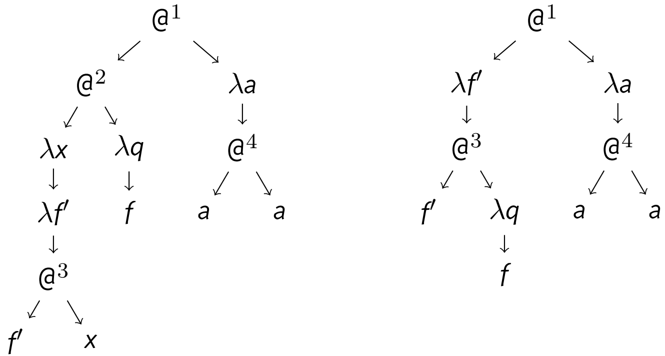
$$(\lambda x f.fx)(\lambda q.f)(\lambda a.aa) \quad \to_{\alpha} \quad (\lambda x f'.f'x)(\lambda q.f)(\lambda a.aa)$$

Solution, Step 2



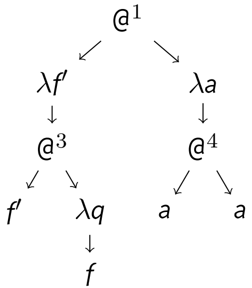
$$(\lambda x f'.f'x)(\lambda q.f)(\lambda a.aa) \qquad \rightarrow_\beta$$

Solution, Step 2



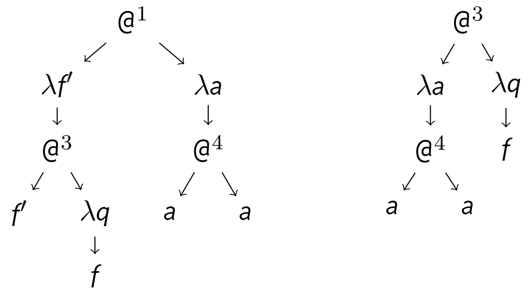
$$(\lambda x f'.f'x)(\lambda q.f)(\lambda a.aa) \quad \to_\beta \quad (\lambda f'.f'(\lambda q.f))(\lambda a.aa)$$

Solution, Step 3



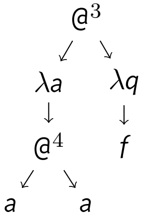
$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \qquad \to_\beta$$

Solution, Step 3



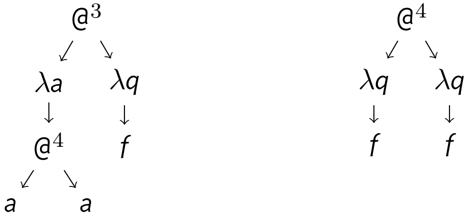
$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \to_\beta \quad (\lambda a.aa)(\lambda q.f)$$

Solution, Step 4



$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \to_\beta$$

Solution, Step 4



$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \rightarrow_\beta \quad (\lambda q.f)(\lambda q.f)$$

Solution, Step 5

$$@^4$$

$$\swarrow \quad \searrow$$

$$\lambda q \qquad \lambda q$$

$$\downarrow \qquad \downarrow$$

$$f \qquad f$$

$$(\lambda q.f)(\lambda q.f) \quad \rightarrow_\beta$$

Solution, Step 5

$$
\begin{array}{ccc}
 & @^{4} & \\
\swarrow & & \searrow \\
\lambda q & & \lambda q \\
\downarrow & & \downarrow \\
f & & f
\end{array}
\qquad\qquad f
$$

$$(\lambda q.f)(\lambda q.f) \quad \to_\beta \quad f$$