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Objectives	Church Numerals	Church Booleans	Arbitrary Data	Objectives	Church Numerals	Church Booleans	Arbitrary Data
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## What Is a Number?

- ► The lambda calculus doesn't have numbers.
- A number n can be thought of as a potential: someday we are going to do something n times.

#### Some Church Numerals

```
1f0 = \langle f-\rangle \langle x-\rangle x

2f1 = \langle f-\rangle \langle x-\rangle f x

3f2 = \langle f-\rangle \langle x-\rangle f (f x)

4f3 = \langle f-\rangle \langle x-\rangle f (f (f x))

1Prelude> let show m = m (+1) 0

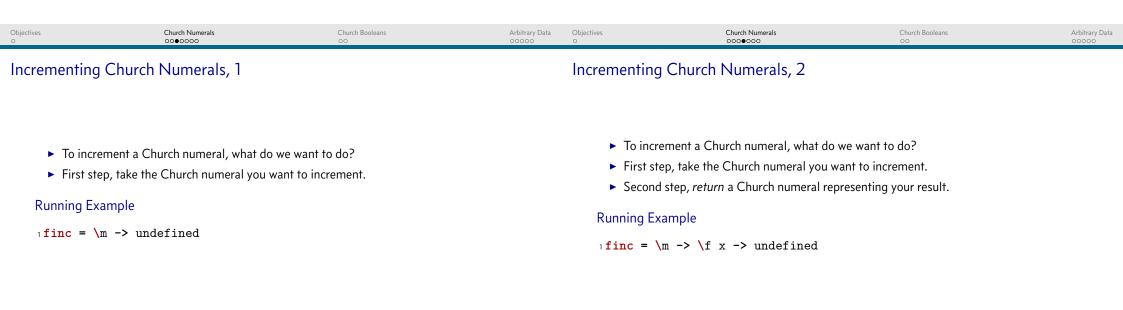
2Prelude> show (\langle f x -\rangle f (f x))

32
```

- Incrementing Church Numerals, 0
  - To increment a Church numeral, what do we want to do?

#### Running Example

#### ifinc = undefined



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Objectives	Church Numerals	Church Booleans	Arbitrary Data	Objectives	Church Numerals	Church Booleans	Arbitrary Data
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## Incrementing Church Numerals, 3

- To increment a Church numeral, what do we want to do?
- First step, take the Church numeral you want to increment.
- Second step, *return* a Church numeral representing your result.
- ► Third step, apply *f* to *x*, *m* times.

### Running Example

 $finc = \ \ -> \ f x -> m f x$ 

# Incrementing Church Numerals, 4

- ▶ To increment a Church numeral, what do we want to do?
- First step, take the Church numeral you want to increment.
- Second step, *return* a Church numeral representing your result.
- ► Third step, apply *f* to *x*, *m* times.
- Finally, apply *f* once more to the result.

### Running Example

 $finc = \ -> \ f x -> f (m f x)$ 

Objectives O	Church Numerals 000000●	Church Booleans 00	Arbitrary Data 00000	Objectives o	Church Numerals 0000000	Church Booleans ●O	Arbitrary Data 00000			
Adding Church	Adding Church Numerals				Implementing Booleans					
<ul> <li>Similar reasoning can yield addition and multiplication.</li> <li>Here is addition. Can you figure our multiplication? Hint: What does (<i>nf</i>) do?</li> <li>Subtraction is a bit more tricky.</li> </ul>				<ul> <li>Church numerals represented integers as a potential number of actions.</li> <li>Church Booleans represent true and false as a choice.</li> <li>T ≡ λab.a F ≡ λab.b</li> </ul>						
Running Exan	Running Example			$1 true = \langle a b - \rangle a$ $2 false = \langle a b - \rangle b$						
$fadd m n = f x \rightarrow m f (n f x)$					= f True False					
					nese into a REPL and try them out! lide: and and or. Try to figure it ou	It before going ahead!				

							· 문 · · · · · · ·
Objectives 0	Church Numerals 0000000	Church Booleans ⊙●	Arbitrary Data 00000	Objectives O	Church Numerals 0000000	Church Booleans OO	Arbitrary Data ●0000
And and Or				Representing Arbitra	ry Types		
<ul> <li>There are a couple</li> </ul>	e of ways to do it.			<ul> <li>Suppose we have</li> </ul>	an algebraic data type with <i>n</i> cons	structors.	

• Each parameter represents one of the constructors.

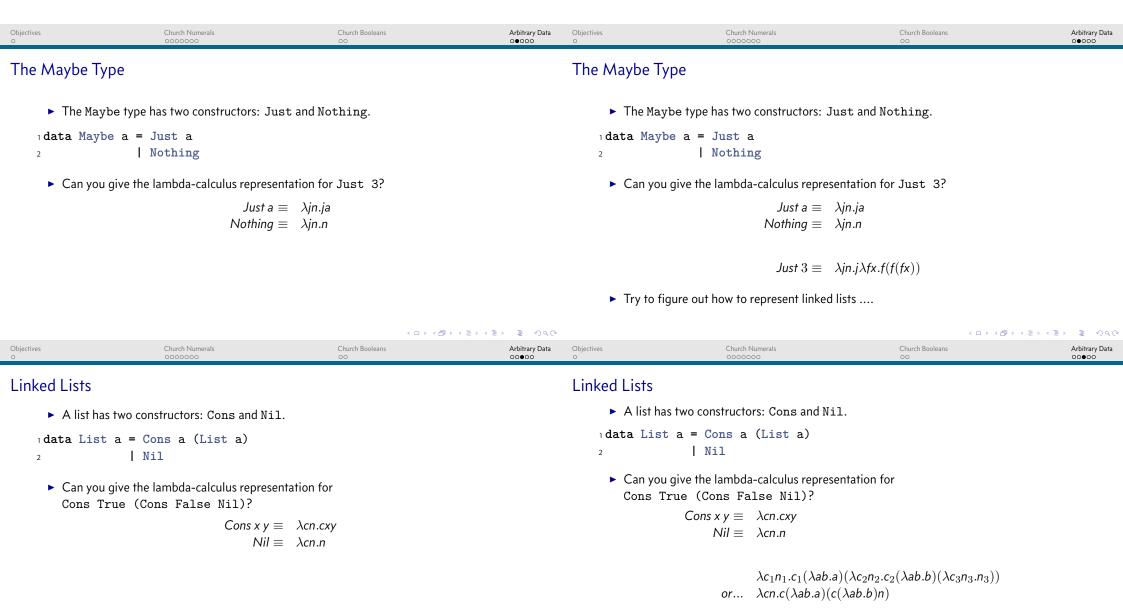
$$T \equiv \lambda ab.a$$
  
 $F \equiv \lambda ab.b$ 

and  $\equiv \lambda xy.xyF$ 

1 and = \x y -> x y false
2 or = \x y -> x true y

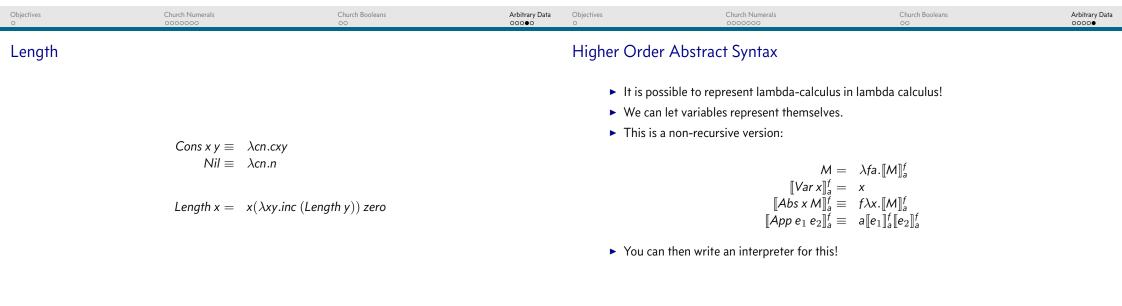
3cif = \c t e -> c t e

 $or \equiv \lambda xy.xTy$ if  $\equiv \lambda cte.cte$ 



Write a function length that determines the length of one of these lists. Assume you are allowed to use recursion. (Note, HASKELL's type system will not let you write this.)

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Objectives 0	Church Numerals	Church Booleans 00	Arbitrary Data 0000●

## Higher Order Abstract Syntax

- ► It is possible to represent lambda-calculus in lambda calculus!
- We can let variables represent themselves.
- ► This is a non-recursive version:

$$\begin{split} \mathsf{M} &= \lambda fa. \llbracket \mathsf{M} \rrbracket_a^f \\ \llbracket \mathsf{Var} \, x \rrbracket_a^f &= x \\ \llbracket \mathsf{Abs} \, x \, \mathsf{M} \rrbracket_a^f &\equiv f \lambda x. \llbracket \mathsf{M} \rrbracket_a^f \\ \llbracket \mathsf{App} \, e_1 \, e_2 \rrbracket_a^f &\equiv a \llbracket e_1 \rrbracket_a^f \llbracket e_2 \rrbracket_a^f \end{split}$$

- > You can then write an interpreter for this!
  - Abstraction: λx.x
  - Application:  $\lambda e_1 e_2 . e_1 e_2$

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