The Y-Combinator

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Objectives

- Use self-application to allow functions to call themselves – even when they don't have names.
- Develop a general combinator Y to implement recursion.

Recursion

Suppose we want to implement

```
f n = f (n+1)
```

Step 1

The outline of the function would look like

$$\lambda n.(f\ (inc\ n))$$

But, how does f get to know itself?

Step 2

Maybe we can tell f by having it take its own name as a parameter.

$$\lambda f.\lambda n.(f\ (inc\ n))$$

So then we pass a copy of f to itself ...

$$(\lambda f.\lambda n.(f\ (inc\ n)))\ (\lambda f.\lambda n.(f\ (inc\ n)))$$

But now f must pass itself into itself ... so we have

$$(\lambda f.\lambda n.((f\ f)\ (inc\ n)))\ (\lambda f.\lambda n.((f\ f)\ (inc\ n)))$$

Expanding a Church Numeral

- Consider how this is similar to the operation of Church numerals.

$$\begin{array}{l} ((f_5\ f)\ x) \\ \to\ (f\ ((f_4\ f)\ x)) \\ \to\ (f\ (f\ ((f_3\ f)\ x))) \\ \to\ (f\ (f\ (f\ ((f_2\ f)\ x)))) \\ \to\ (f\ (f\ (f\ (f\ (f\ x))))) \end{array}$$

So ...

$$((f_n\ f)\ x)\ \to\ (f\ ((f_{n-1}\ f)\ x))$$

What would it look like to have an f_∞?

The Y-Combinator

Consider this pattern:

$$(f_\infty\ f)\ x\ \to\ f\ (f_\infty\ f)\ x$$

- What can you tell about f? About f_∞?
- Definition: combinator = higher order function that produces its result only though function application.
- The problem with the above function is that there's no way out. How can we stop the function when we are done?

Coding the Y-Combinator

$$(Y\ f)\ \to\ f\ (Y\ f)$$

So...

$$Y\ =\ \lambda f.(\lambda y.f\ (y\ y))\ \lambda y.f\ (y\ y))$$

The function f must take $(Y\ f)$ as an argument.

$$\begin{array}{rcl} (Y\ F) & = & (\lambda f.(\lambda y.f\ (y\ y))\ \lambda y.f\ (y\ y))\ F \\ & = & (\lambda y.F\ (y\ y))\ \lambda y.F\ (y\ y) \\ & = & F\ ((\lambda y.F\ (y\ y))\lambda y.F\ (y\ y)) \\ & = & F\ (Y\ F) \end{array}$$

Example

```
fact n =
  if n < 1 then 1
           else n * (fact (n-1))
```

In λ-calculus:

$$\begin{array}{l}\lambda f.\lambda n.\\ \quad \texttt{if}\ n < 1\ \texttt{then}\ 1\\ \quad\quad \texttt{else}\ n * (f\ (n-1))\end{array}$$

Then we have:

$$Y\ fact \rightarrow \begin{array}{l}\lambda n.\\ \quad \texttt{if}\ n < 1\ \texttt{then}\ 1\\ \quad\quad \texttt{else}\ n * ((Y\ fact)\ (n-1))\end{array}$$

Further Reading

- You can use λ-calculus to represent itself using these techniques. You already have everything you need to do it. You can see the details in Torben Æ. Mogensen's paper, "Efficient Self-Interpretations in Lambda Calculus," in the *Journal of Functional Programming* v2 n3.