# Unification

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# Objectives

## You should be able to ...

Unification is a third major topic that will appear many times in this course. It is used in languages such as HASKELL and PROLOG, and also in theoretical discussions.

- Describe the problem that unification solves.
- Solve a unification problem.
- Implement unification in HASKELL.
- Describe some use cases for unification.

# The Domain

**Terms** Have *name* and *arity*

- The name will be in western alphabet.
- Arity = “number of arguments” – may be zero
- Examples: `x`, `z`, `f(x,y)`, `x(y,f,z)`

**Variables** Written using Greek alphabet, may be subscripted

- Represent a target for substitution
- Examples: $\alpha, \beta_{12}, \gamma_7$

**Substitutions** Mappings from variables to terms

- Examples: $\sigma = \{\alpha \mapsto \mathtt{f}(\mathtt{x}, \beta), \beta \mapsto \mathtt{y}\}$
- Substitutions are *applied*: $\sigma(\mathtt{g}(\beta)) \rightarrow \mathtt{g}(\mathtt{y})$

Note: arguments to terms may have non-zero arity, or may be variables.

# The Problem

- Given terms $s$ and $t$, try to find a substitution $\sigma$ such that $\sigma(s) = \sigma(t)$.
- If such a substitution exists, it is said that $s$ and $t$ unify.
- A *unification problem* is a set of equations $S = \{s_1 = t_1, s_2 = t_2, \ldots\}$.
- A unification problem $S = \{x_1 = t_1, x_2 = t_2, \ldots\}$ is in *solved form* if
  - The terms $x_i$ are distinct variables.
  - None of them occur in $t_i$.

Our approach: given a unification problem $S$, we want to find the most general unifier $\sigma$ that solves it. We will do this by transforming the equations.

# Four Operations

Start with a unification problem $S = \{s_1 = t_1, s_2 = t_2, \ldots\}$ and apply the following transformations as necessary:

| | |
|---|---|
| Delete | A trivial equation $t = t$ can be deleted. |
| Decompose | An equation $f(\overline{t_n}) = f(\overline{u_n})$ can be replaced by the set $\{t_1 = u_1, \ldots, t_n = u_n\}$. |
| Orient | An equation $t = x$ can be replaced by $x = t$ if $x$ is a variable and $t$ is not. |
| Eliminate | an equation $x = t$ can be used to substitute all occurrences of $x$ in the remainder of $S$. |

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$\{\alpha = f(x),\ \beta = f(x)\}$

Now we are done ....

$S = \{\alpha \mapsto f(x),\ \beta \mapsto f(x)\}$

# Unification Failures

There are two situations that can cause unification to fail:

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2. Failing the "occurs check"

$$f(\alpha) = f(f(\alpha))$$

# Implementation

To implement this in a programming language:

- Keep two lists: one for the incoming equations, one for the solved variables.
- Remove the first element of the incoming list.
  - Decompose and delete manipulate the incoming list.
  - Orient and eliminate can be handled in one case.
- Your solution list contains the result once the incoming list is empty.

# Example – Compatibility

- Your advisor wants you to take CS 421 and some theory class.
- Your mom wants you to take CS 374 and some language class.
- Can both your advisor and your mom be happy?

This is a problem we can solve using unification:

- Let $f$ be a "schedule function," the first argument is a language class, the second argument is a theory class.
- $s = f(cs421, \beta)$ (where $\beta$ is a theory class)
- $t = f(\alpha, cs374)$ (where $\alpha$ is a language class)
- Let $\sigma = \{\alpha \mapsto cs421, \quad \beta \mapsto cs374\}$

# Example – Types

Type checking is also a form of unification.

```
map :: (a -> b) -> [a] -> [b]
inc :: Int -> Int
foo :: [Int]
```

Will `map(inc)(foo)` work?

$$S = \{(\alpha \Rightarrow \beta) = (\mathtt{Int} \Rightarrow \mathtt{Int}), \quad \mathtt{List}[\alpha] = \mathtt{List}[\mathtt{Int}]\}$$

# Type Checking Solution

$$S = \{(\alpha \Rightarrow \beta) = (\mathtt{Int} \Rightarrow \mathtt{Int}), \quad \mathtt{List}[\alpha] = \mathtt{List}[\mathtt{Int}]\}$$

- Decompose: $\{\alpha = \mathtt{Int}, \quad \beta = \mathtt{Int}, \quad \mathtt{List}[\alpha] = \mathtt{List}[\mathtt{Int}]\}$
- Substitute: $\{\alpha = \mathtt{Int}, \quad \beta = \mathtt{Int}, \quad \mathtt{List}[\mathtt{Int}] = \mathtt{List}[\mathtt{Int}]\}$
- Delete: $\{\alpha = \mathtt{Int}, \quad \beta = \mathtt{Int}\}$

The original type of `map` was $(\alpha \Rightarrow \beta) \Rightarrow \mathtt{List}[\alpha] \Rightarrow \mathtt{List}[\beta]$.
We can use our pattern to get the output type: $S(\mathtt{List}[\beta]) \equiv \mathtt{List}[\mathtt{Int}]$.

# Example 2 – Types

Here's an example that fails.

```
map :: (a->b) -> [a] -> [b]
inc : String -> Int
foo : [Int]
```

Will `map(inc)(foo)` work?

$$S = \{(\alpha \Rightarrow \beta) = (\texttt{String} \Rightarrow \texttt{Int}), \quad \texttt{List}[\alpha] = \texttt{List}[\texttt{Int}]\}$$

# Type Checking 2 Solution

$$S = \{(\alpha \Rightarrow \beta) = (\texttt{String} \Rightarrow \texttt{Int}), \quad \texttt{List}[\alpha] = \texttt{List}[\texttt{Int}]\}$$

- Decompose: $\{\alpha = \texttt{String}, \quad \beta = \texttt{Int}, \quad \texttt{List}[\alpha] = \texttt{List}[\texttt{Int}]\}$
- Substitute: $\{\alpha = \texttt{string}, \quad \beta = \texttt{Int}, \quad \texttt{List}[\texttt{String}] = \texttt{List}[\texttt{Int}]\}$
- Error: $\texttt{List}[\texttt{string}] \neq \texttt{List}[\texttt{Int}]$!